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Jackson Theorems in Hardy Spaces and Approximation by Riesz Means

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An analog of the theorem of D. Jackson on the approximation of periodic functions by means of trigonometric polynomials is established for some Hardy spaces of several variables. © 1987 Academic Press, Inc.

An important result in the theory of approximation is the theorem of D. Jackson:

If f is a 2π-periodic function on R with continuous derivatives up to the kth order, then for every $n>0$ there exists a trigonometric polynomial P of degree at most n, such that

$$\operatorname*{Sup}_{x\in R} |f(x)-P(x)| \leqslant cn^{-k} \operatorname*{Sup}_{|h|\leqslant 1/n} \operatorname*{Sup}_{x\in R} \left|\frac{d^k}{dx^k} f(x+h) - \frac{d^k}{dx^k} f(x)\right|.$$

This theorem has been generalized and extended to various classes of functions and moduli of continuity of arbitrary order (see, for example, [4]). In particular, Storoženko has proved an analog of the above inequality for the classical Hardy spaces on the unit disc of the complex plane (see [12, 13]). In this paper we are concerned with an extension of Jackson's theorem to other Hardy spaces of several variables. Our main interest is in approximating distributions in $H^p(R^N)$, $0<p<+\infty$, by means of entire functions of finite exponential type (the analog in R^N of the trigonometric polynomials). However, we shall be able to prove a theorem of Jackson type in $H^p(R^N)$ only after having established an analog of this theorem for Hardy spaces on polydiscs and on some tube domains of C^N.

Section 1 of this paper contains the notation and some preliminary material which is also of some independent interest. In particular, we establish a Poisson type summation formula between Hardy spaces on the polydisc $U^N=\{z\in C^N\colon |z_j|<1\}$, and Hardy spaces on the poly-half-space $D^N=\{z\in C^N\colon \text{Im. } z_j>0\}$. In Sections 2, 3, and 4 we state a theorem of

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Jackson type for Hardy spaces on U^N, D^N, and R^N, respectively. The proof of this theorem in $H^p(U^N)$ is essentially that of Storoženko for $H^p(U)$. The proof in $H^p(D^N)$ and $H^p(R^N)$ follows from the result in $H^p(U^N)$ and some transference methods. Finally, in Section 5 we study the approximation properties of the Riesz means in Hardy spaces.

1. Hardy Spaces of Several Variables

Let $U^N = \{z = (z_1, ..., z_N) \in C^N : |z_j| < 1\}$ be the unit polydisc in C^N, and let $T^N = \{z = (z_1, ..., z_N) \in C^N : |z_j| = 1\}$ be its distinguished boundary, the N-dimensional torus.

The Hardy space $H^p(U^N)$, $0 < p \leqslant +\infty$, is the set of all holomorphic functions f on U^N for which

$$\|f\|_{H^p(U^N)} = \operatorname*{Sup}_{0 \leqslant r < 1} \left(\int_{T^N} |f(rz)|^p \, dz \right)^{1/p} < +\infty$$

(see [7, 8]).

The polydisc is closely related to another domain in C^N. Let $D^N = \{z = (z_1, ..., z_N) \in C^N : \text{Im. } z_j > 0\}$ be the poly-half-space in C^N. If $(R_+)^N$ denotes the set $\{y = (y_1, ..., y_N) \in R^N : y_j > 0\}$, then D^N is the tube domain $R^N + i(R_+)^N$.

The Hardy space $H^p(D^N)$, $0 < p \leqslant +\infty$, is the set of all holomorphic functions f on D^N for which

$$\|f\|_{H^p(D^N)} = \operatorname*{Sup}_{y \in (R_+)^N} \left(\int_{R^N} |f(x + iy)|^p \, dx \right)^{1/p} < +\infty$$

(see [11]).

Finally, we also introduce a "real" Hardy space. The Hardy space $H^p(R^N)$, $0 < p < +\infty$, is the set of all harmonic functions f on $R_+^{N+1} = R^N \times R_+$, for which

$$\|f\|_{H^p(R^N)} = \left(\int_{R^N} |\operatorname*{Sup}_{y \in R_+} |f(x, y)||^p \, dx \right)^{1/p} < +\infty$$

(see [3, 9, and 11]).

Although defined on different domains, the three Hardy spaces $H^p(U^N)$, $H^p(D^N)$, and $H^p(R^N)$, are closely related and share many important properties. Perhaps the best way to see this is to look at the Fourier transform. Let us start by considering the relation between $H^p(R^N)$ and $H^p(D^N)$.

If f is in $H^p(R^N)$ (resp. $H^p(D^N)$), then $\text{Lim}_{y \to 0} f(\cdot, y)$ (resp. $\text{Lim}_{y \to 0} f(\cdot + iy)$) exists in the topology of tempered distributions on R^N. Let us denote this limit by the same letter f, and let us denote by $\hat{f}$ the Fourier transform of this tempered distribution. In the sequel we shall make no distinction between a function f and the associated (boundary value) tempered distribution f. The support of the Fourier transform of a distribution in $H^p(R^N)$ can be all of R^N. On the contrary, if this distribution is in $H^p(D^N)$, then its Fourier transform has support contained in $\overline{(R_+)}^N$. However, apart from this important difference, most of the properties of the Fourier transform of distributions in $H^p(R^N)$ are also valid for the Fourier transform of distributions in $H^p(D^N)$. This is a consequence of the following theorem of Carleson [2].

THEOREM 1.1: *Let f be in $H^p(D^N)$, $0<p<+\infty$. Then f is also in $H^p(R^N)$, and $\|f\|_{H^p(R^N)} \leqslant c\|f\|_{H^p(D^N)}$. Vice versa, let $\{m_j\}$ be a finite C^∞-partition of unity of $R^N-\{0\}$, such that every m_j is homogeneous of degree zero and has support contained in a rotation of $\overline{(R_+)}^N$. Then, every f in $H^p(R^N)$ can be decomposed as $f=\sum_j f_j$, where $\hat{f}_j=m_j\hat{f}$. If $\sigma_j \in SO(N)$ is suitably chosen, $f_j \circ \sigma_j$ is in $H^p(D^N)$, and $\|f_j \circ \sigma_j\|_{H^p(D^N)} \leqslant c\|f\|_{H^p(R^N)}$.*

An immediate consequence of this theorem is the fact that every multiplier operator bounded on $H^p(R^N)$ is automatically bounded also on $H^p(D^N)$. Moreover, the decomposition of (a tempered distribution) f in $H^p(R^N)$ into a sum of (boundary values of) functions holomorphic in tube domains in C^N expressed by this theorem preserves the smoothness properties of f. We shall make use of these observations in the sequel.

So much for the spaces $H^p(R^N)$ and $H^p(D^N)$. We want to consider now the relation between $H^p(D^N)$ and $H^p(U^N)$. The case of interest for us is when $0<p\leqslant 1$. In this case a Poisson type summation formula between these two spaces holds.

Let $Q^N=\{x=(x_1,..., x_N) \in R^N\colon -1/2 \leqslant x_j < \frac{1}{2}\}$ be the unit cube in R^N. Then Q^N can be naturally identified with the torus T^N via the map $(x_1,..., x_N) \to (e^{2\pi i x_1},..., e^{2\pi i x_N})$, and the Lebesgue measure dx on Q^N corresponds to the normalized Haar measure dz on T^N.

THEOREM 1.2. *If f is in $H^p(D^N)$, $0<p\leqslant 1$, define $\tilde{f}(z)=\sum_k \hat{f}(k)\, z^k$ ($k=(k_1,..., k_N)$ is a multi-index, and $z^k=z_1^{k_1}\cdot \cdots \cdot z_N^{k_N}$). Then $\tilde{f}$ is in $H^p(U^N)$, and $\|\tilde{f}\|_{H^p(U^N)} \leqslant \|f\|_{H^p(D^N)}$. Vice versa, if t is a positive real number, and $f_t(z)=t^{-N}f(t^{-1}z)$, then f_t is in $H^p(D^N)$, and $\operatorname{Lim}_{t\to 0} t^{N(1-1/p)}\|\tilde{f}_t\|_{H^p(U^N)} = \|f\|_{H^p(D^N)}$.*

The inequality $\|\tilde{f}\|_{H^p(U^N)} \leqslant \|f\|_{H^p(D^N)}$ generalizes the classical Poisson summation formula between Fourier series and integrals. The converse identity $\operatorname{Lim}_{t\to 0} t^{N(1-1/p)}\|\tilde{f}_t\|_{H^p(U^N)} = \|f\|_{H^p(D^N)}$ is in some sense motivated by the following heuristic argument: The Fourier transform of f_t is $\hat{f}(t\,\cdot\,)$. Hence $\tilde{f}_t(e^{2\pi i x_1},..., e^{2\pi i x_N})=\sum_k \hat{f}(tk)\, e^{2\pi i tk\cdot t^{-1}x}$, and if t is small, this sum is similar to the integral $t^{-N}\int_{(R_+)^N} \hat{f}(\xi)\, e^{2\pi i\xi\cdot t^{-1}x} d\xi = f_t(x)$. But $\|f_t\|_{H^p(D^N)} = t^{N(1/p-1)}\|f\|_{H^p(D^N)}$!

Proof of the theorem. First notice that since f is in $H^p(D^N)$ with $0 < p \leqslant 1$, the Fourier transform $\hat{f}$ of f is a continuous function supported in $\overline{(R_+)}^N$, and $\hat{f}$ has at most a polynomial growth: $|\hat{f}(\xi)| \leqslant c\|f\|_{H^p(D^N)}\, |\xi|^{N(1/p-1)}$ (see [11, 14], and Theorem 1.1). This implies that $\tilde{f}$ is well defined, and is a holomorphic function on U^N. Let $z = (z_1, ..., z_N) \in U^N$, and, if every z_j is different from zero, write $z_j = e^{2\pi i(x_j + iy_j)}$, with $x \in Q^N$ and $y \in (R_+)^N$. Then, since the function $f(\cdot + iy)$ is in $L^1(R^N)$, and $(f(\cdot + iy))\hat{}(\xi) = \hat{f}(\xi)\, e^{-2\pi y\cdot\xi}$ (see [11]), we have $\tilde{f}(z) = \sum_k f(x + iy + k)$, by the classical Poisson summation formula. Hence,

$$\int_{Q^N} |\tilde{f}(e^{2\pi i(x_1 + iy_1)}, ..., e^{2\pi i(x_N + iy_N)})|^p\, dx$$

$$\leqslant \sum_k \int_{Q^N} |f(x + iy + k)|^p\, dx = \int_{R^N} |f(x + iy)|^p\, dx.$$

Taking the supremum with respect to $y \in (R_+)^N$ on both sides of this inequality we obtain $\|\tilde{f}\|_{H^p(U^N)} \leqslant \|f\|_{H^p(D^N)}$.

Let us prove the converse. Let $f(x) = \mathrm{Lim}_{y\to 0} f(x + iy)$. Then $\mathrm{Lim}_{r\to 1} \tilde{f}_t(r(e^{2\pi i x_1}, ..., e^{2\pi i x_N})) = t^{-N} \sum_k f(t^{-1}x + t^{-1}k)$ (this series is almost everywhere absolutely convergent), and to prove that $\mathrm{Lim}_{t\to 0} t^{N(1-1/p)}\|\tilde{f}_t\|_{H^p(U^N)} = \|f\|_{H^p(D^N)}$ it is enough to show that

$$\operatorname*{Lim}_{t\to 0} t^{-N} \int_{Q^N} \left| \sum_k f(t^{-1}x + t^{-1}k) \right|^p dx = \int_{R^N} |f(x)|^p\, dx.$$

But

$$|f(t^{-1}x)|^p - \sum_{k\neq 0} |f(t^{-1}x + t^{-1}k)|^p$$

$$\leqslant \left| \sum_k f(t^{-1}x + t^{-1}k) \right|^p$$

$$\leqslant |f(t^{-1}x)|^p + \sum_{k\neq 0} |f(t^{-1}x + t^{-1}k)|^p,$$

and since

$$t^{-N} \int_{Q^N} |f(t^{-1}x)|^p\, dx = \int_{t^{-1}Q^N} |f(x)|^p\, dx$$

$$\to \int_{R^N} |f(x)|^p\, dx \qquad \text{as } t \to 0,$$

and

$$t^{-N} \int_{Q^N} \sum_{k \neq 0} |f(t^{-1}x + t^{-1}k)|^p \, dx \leqslant \int_{|x| > t^{-1}/2} |f(x)|^p \, dx$$
$$\to 0 \qquad \text{as } t \to 0,$$

the desired result follows. ∎

2. The Jackson Theorem for the Polydisc

Let f be a function in $H^p(U^N)$. The best approximation of f by polynomials of degree at most n in the H^p-metric is defined by

$$E(n, f, H^p(U^N)) = \operatorname*{Inf}_{p} \|f - p\|_{H^p(U^N)},$$

where $p(z) = \sum_{|k| \leqslant n} \hat{p}(k)\, z^k$ is a polynomial of degree at most n. Let m be a nonnegative integer, and let t and u be real numbers, with $t > 0$. Define

$$\Delta_u^m f(z) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(e^{2\pi iju} z)$$

and

$$\omega_m(t, f, H^p(U^N)) = \operatorname*{Sup}_{|u| \leqslant N^{-1/2} t} \|\Delta_u^m f\|_{H^p(U^N)}.$$

THEOREM 2.1. *If f is in $H^p(U^N)$, $0 < p \leqslant +\infty$, then for every positive integer n we have $E(n, f, H^p(U^N)) \leqslant c\,\omega_m(1/n, f, H^p(U^N))$.*

The case $p \geqslant 1$ of this theorem is well known (see [4]), and the case $p < 1$ is due to Storoženko (see [12, 13] for $N = 1$, and [16] for arbitrary N). Actually, our proof of this theorem is a simple extension of the 1-dimensional proof in [13]; we reduce the problem to one dimension by means of the slices of the function f.

Sketch of the proof. Let $\alpha > -1$, and define

$$P(z) = \frac{\Gamma(n+1)\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \times \sum_{j=1}^{m} (-1)^j \binom{m}{j} \int_{-1/2}^{1/2} f((re^{2\pi iu})^j z) \left(\frac{1 - (re^{2\pi iu})^{n+1}}{1 - re^{2\pi iu}} \right)^{\alpha+1} (re^{2\pi iu})^{-n} \, du.$$

Then it is possible to verify that P is a polynomial of degree at most n, and, if α is big enough and $r = 1 - 1/n$,

$$\|f - P\|_{H^p(U^N)} \leqslant c\omega_m(1/n, f, H^p(U^N)).$$

The details are as in [13]. ∎

COROLLARY 2.2. *Let ϕ be a C^∞-function on R^N with compact support, and such that $\phi(\xi) = 1$ if $|\xi| \leqslant 1$. If $f(z) = \sum_k \hat{f}(k) z^k$ and $s > 0$, define*

$$\Phi_s * f(z) = \sum_k \phi(sk) \hat{f}(k) z^k.$$

Suppose that f is in $H^p(U^N)$. Then also $\Phi_s * f$ is in $H^p(U^N)$, and

$$\|f - \Phi_s * f\|_{H^p(U^N)} \leqslant c\omega_m(s, f, H^p(U^N)).$$

Obviously this corollary holds under more general conditions on the function ϕ. However in this case the proof is immediate, and in the next section we shall need only this weaker result.

Remark. We stated the theorem for the polydisc, but the same proof holds for other bounded balanced domains D in C^N (a domain D in C^N is called balanced if $wz \in D$ whenever $z \in D$, $w \in C$, and $|w| \leqslant 1$). It is also interesting to notice that although the distinguished boundary of D has real dimension at least N, in the proof of the theorem we used only difference operators along the direction determined by the slices of D, $D_z = \{wz \in C^N: w \in C, |w| \leqslant 1\}$. In this context see also [8].

3. THE JACKSON THEOREM FOR THE POLY-HALF-SPACE

Let f be a function in $H^p(D^N)$. The best approximation of f by entire functions of exponential type at most s is defined by

$$E(s, f, H^p(D^N)) = \operatorname{Inf}_g \|f - g\|_{H^p(D^N)},$$

where g is an entire function of exponential type at most s. Let m be a non-negative integer, t be a positive real number, and let h be a vector in R^N with all entries equal, i.e., $h = u(1, \ldots, 1)$ for some real number u. Define

$$\Delta_h^m f(z) = \sum_{j=0}^{m} (-)^{m-j} \binom{m}{j} f(z + jh),$$

and

$$\omega_m(t, f, H^p(D^N)) = \operatorname{Sup}_{|h| \leqslant t} \|\Delta_h^m f\|_{H^p(D^N)}.$$

THEOREM 3.1. *If f is in $H^p(D^N)$, $0<p\leqslant+\infty$, then for every $s>0$ we have* $E(s^{-1}, f, H^p(D^N))\leqslant c\omega_m(s, f, H^p(D^N))$.

Again, the case $p\geqslant 1$ of this theorem is well known (see [4, 6]), so that we shall consider only the case $0<p\leqslant 1$. We first need a couple of easy lemmas. We recall that $f_t(z)=t^{-N}f(t^{-1}z)$ and $\tilde{f}(z)=\sum_k \hat{f}(k)\,z^k$.

LEMMA 3.2. $\omega_m(s, f_t, H^p(D^N))=t^{N(1/p-1)}\omega_m(t^{-1}s, f, H^p(D^N))$.

Proof. This lemma is an immediate consequence of the identity $\|f_t\|_{H^p(D^N)}=t^{N(1/p-1)}\|f\|_{H^p(D^N)}$.

LEMMA 3.3. *If $h=(1,...,1)\in R^N$, and s, u are real numbers, with $s>0$, then* $(\Delta^m_{uh}f)^{\sim}=\Delta^m_u\tilde{f}$ and $\omega_m(s, \tilde{f}, H^p(U^N))\leqslant\omega_m(s, f, H^p(D^N))$.

Proof. The identity $(\Delta^m_{uh}f)^{\sim}=\Delta^m_u\tilde{f}$ is an immediate consequence of the equations $(\Delta^m_h f)^{\wedge}(\xi)=(e^{2\pi i h\cdot\xi}-1)^m\hat{f}(\xi)$ and $\Delta^m_u z^k=(e^{2\pi i u|k|}-1)^m z^k$. The inequality $\omega_m(s, \tilde{f}, H^p(U^N))\leqslant\omega_m(s, f, H^p(D^N))$ is an immediate consequence of the equation $(\Delta^m_{uh}f)^{\sim}=\Delta^m_u\tilde{f}$ and Theorem 1.2.

Proof of the Theorem. As we said before, we consider only the case $0<p\leqslant 1$. Let ϕ be a C^∞-function on R^N, with support in $\{\xi\in R^N: |\xi|\leqslant 2\}$, and such that $\phi(\xi)=1$ if $|\xi|\leqslant 1$. If f is in $H^p(D^N)$ define

$$\Phi_s * f(z)=\int_{R^N}\phi(s\xi)\,\hat{f}(\xi)\,e^{2\pi i\xi\cdot z}\,d\xi,$$

so that $(\Phi_s * f)^{\wedge}(\xi)=\phi(s\xi)\hat{f}(\xi)$. Then $\Phi_s * f$ is an entire function of exponential type at most $2s^{-1}$, and, by Corollary 2.2 and Lemmas 3.2 and 3.3, for every $t>0$ we have

$$\begin{aligned}\|\tilde{f}_t-(\Phi_{st} * f_t)^{\sim}\|_{H^p(U^N)}&\leqslant c\omega_m(st, \tilde{f}_t, H^p(U^N))\\&\leqslant c\omega_m(st, f_t, H^p(D^N))\\&=ct^{N(1/p-1)}\omega_m(s, f, H^p(D^N)).\end{aligned}$$

From this inequality and Theorem 1.2 we obtain

$$\begin{aligned}\|f-\Phi_s * f\|_{H^p(D^N)}&=\operatorname*{Lim}_{t\to 0} t^{N(1-1/p)}\|\tilde{f}_t-(\Phi_{st} * f_t)^{\sim}\|_{H^p(U^N)}\\&\leqslant c\omega_m(s, f, H^p(D^N)),\end{aligned}$$

and this proves the theorem. ∎

We conclude this section by stating a second theorem of Jackson type concerning the relation between the best approximation of functions in

$H^p(D^N)$ by entire functions of finite exponentional type and the moduli of continuity of the derivatives of such functions.

Let α be a real number. The Bessel potential $I^\alpha f$ of a tempered distribution f is defined via the Fourier transform by $(I^\alpha f)\hat{}(\xi) = (1-|\xi|^2)^{-\alpha/2}\hat{f}(\xi)$. When $\alpha > 0$, $I^{-\alpha}f$ is a sort of fractional derivative of f, and indeed one can pass from the Bessel potential to the derivatives via "nice" multiplier transformations (for example see [4 or 9]).

THEOREM 3.4. *If $\alpha \geqslant 0$, and if $I^{-\alpha}f$ is in $H^p(D^N)$, $0 < p \leqslant +\infty$, then also f is in $H^p(D^N)$ and $E(s^{-1}, f, H^p(D^N)) \leqslant cs^\alpha\, \omega_m(s, I^{-\alpha}f, H^p(D^N))$.*

We notice that Theorem 3.1 is a particular case ($\alpha = 0$) of this theorem. Actually it is also possible to prove an analog of this theorem with the derivatives $\{\partial^k/\partial z^k f\}$ instead of the Bessel potential $I^{-\alpha}f$.

Proof of the Theorem. Let ϕ and $\Phi_s * f$ be defined as in Theorem 3.1. Then since the support of $(1-\phi(s\cdot))$ is contained in $\{\xi \in R^N: |\xi| \geqslant s^{-1}\}$ and $(1-\phi(2s\xi)) = 1$ if $|\xi| \geqslant s^{-1}$, we have

$$\begin{aligned}(f - \Phi_s * f)\hat{}(\xi) &= (1-\phi(s\xi))\,\hat{f}(\xi)\\ &= (1-\phi(2s\xi))(1+|\xi|^2)^{-\alpha/2}(1-\phi(s\xi))(1+|\xi|^2)^{\alpha/2}\,\hat{f}(\xi)\\ &= (1-\phi(2s\xi))(1+|\xi|^2)^{-\alpha/2}(I^{-\alpha}f - \Phi_s * I^{-\alpha}f)\hat{}(\xi).\end{aligned}$$

Now, the operator norm of the multiplier $(1-\phi(2s\cdot))(1+|\cdot|^2)^{-\alpha/2}$, acting on $H^p(D^N)$, is dominated by cs^α. This follows from the study of the kernel associated with the operator I^α (see [4,9]), or from the Hörmander multiplier theorem for Hardy spaces (see [9, 14]). Hence, by Theorem 3.1,

$$\begin{aligned}\|f - \Phi_s * f\|_{H^p(D^N)} &\leqslant cs^\alpha \|I^{-\alpha}f - \Phi_s * I^{-\alpha}f\|_{H^p(D^N)}\\ &\leqslant cs^\alpha\, \omega_m(s, I^{-\alpha}f, H^p(D^N)).\end{aligned}$$

Since $\Phi_s * f$ is an entire function of exponential type at most $2s^{-1}$, the theorem follows. ∎

4. THE JACKSON THEOREM FOR R^N

Let f be a function in $H^p(R^N)$. The best approximation of f by entire functions of exponential type at most s is defined by

$$E(s, f, H^p(R^N)) = \operatorname*{Inf}_g \|f - g\|_{H^p(R^N)},$$

where g is an entire function of exponential type at most s. Let m be a non-negative integer, t be a positive real number, and let h be a vector in R^N.

Define

$$\Delta_h^m f(x, y) = \sum_{j=0}^{m} (-)^{m-j} \binom{m}{j} f(x+jh, y)$$

(x is the variable in R^N, and y the one in R_+), and

$$\omega_m(t, f, H^p(R^N)) = \operatorname*{Sup}_{|h| \leqslant t} \|\Delta_h^m f\|_{H^p(R^N)}.$$

THEOREM 4.1. *If $\alpha \geqslant 0$, and if $I^{-\alpha} f$ is in $H^p(R^N)$, $0 < p < +\infty$, then also f is in $H^p(R^N)$, and $E(s^{-1}, f, H^p(R^N)) \leqslant cs^\alpha \omega_m(s, I^{-\alpha} f, H^p(R^N))$.*

Proof. As in Theorem 1.1, decompose f as $f = \sum_j f_j$. Then for every j and s we have $\omega_m(s, I^{-\alpha} f_j, H^p(R^N)) \leqslant c\omega_m(s, I^{-\alpha} f, H^p(R^N))$. Since after a suitable rotation $I^{-\alpha} f_j$ is in $H^p(D^N)$, the theorem follows from Theorem 3.4. ∎

Remark. By means of Theorem 4.1 and the Hörmander multiplier theorem for Hardy spaces it is not difficult to prove that if f is in $H^p(R^N)$, then

$$\omega_m(t, f, H^p(R^N)) \cong \operatorname*{Inf}_g (\|f-g\|_{H^p(R^N)} + t^m \| I^{-m} g\|_{H^p(R^N)}) \tag{4.2}$$

and

$$\operatorname*{Inf}_g (\|f-g\|_{H^p(R^N)} + t^m \| I^{-m} g\|_{H^p(R^N)}) \tag{4.3}$$

$$\cong t^m \left\| \frac{d^m}{dt^m} f(\cdot, t) \right\|_{H^p(R^N)},$$

where g varies in $H^p(R^N)$. The estimates (4.2) and (4.3) give two explicit characterizations of the K-functional between the spaces $H^p(R^N)$ and $I^{-m} H^p(R^N)$. This is of some importance in the study of Besov–Lipschitz spaces defined by means of the H^p-metric. In particular, using Theorem 4.1 and estimates (4.2) and (4.3) it is possible to prove that if $0 < \theta < 1$ and $0 < q \leqslant +\infty$, the quasi-norms

(i) $\|f\|_{H^p(R^N)} + \left(\int_0^{+\infty} |t^{-m\theta} \omega_m(t, f, H^p(R^N))|^q \, dt/t \right)^{1/q}$,

(ii) $\|f\|_{H^p(R^N)} + \left(\int_0^{+\infty} |s^{m\theta} E(s, f, H^p(R^N))|^q \, ds/s \right)^{1/q}$,

(iii) $\|f\|_{H^p(R^N)} + \left(\int_0^{+\infty} |y^{m(1-\theta)} \|d^m/dy^m f(\cdot, y)\|_{H^p(R^N)}|^q \, dy/y \right)^{1/q}$, are equivalent, and define the Besov–Lipschitz space $(H^p(R^N), I^{-m}H^p(R^N))_{\theta,q}$, i.e., the interpolation space, in the real method of interpolation, between $H^p(R^N)$ and $I^{-m}H^p(R^N)$ (see also [4, 5, 6, 9, and 15]).

5. Approximation by Riesz Means

Let δ be a complex number with positive real part, and let m be a positive integer. If f is in $H^p(R^N)$, $0<p<+\infty$, the Riesz means $\{R_s^{\delta,m}*f\}$ of order δ and type m are defined by

$$(R_s^{\delta,m}*f)\hat{}(\xi)=(1-|s\xi|^m)_+^\delta \hat{f}(\xi)$$

$((1-|\xi|^m)_+=\operatorname{Max}\{0,(1-|\xi|^m)\})$. When $m=2$ these means are also called the Bochner–Riesz means, and in this case it is well known that if $\operatorname{Re}\delta$ is big enough, $\operatorname{Re}\delta>\delta(p)$, then these means are bounded on $H^p(R^N)$. The "critical index" $\delta(p)$ is $N/p-(N+1)/2$ when $0<p\leqslant 1$, and it is at most $(N-1)|1/p-1/2|$ when $1<p<+\infty$ (see [10, 11]. Actually, when $1<p<+\infty$ more precise results are known). It turns out that

$$(1-|\xi|^m)_+^\delta=\left(\frac{1+|\xi|+\cdots+|\xi|^{m-1}}{1+|\xi|}\right)^\delta(1-|\xi|^2)_+^\delta,$$

and then by the Hörmander multiplier theorem for Hardy spaces, for every m and every δ with $\operatorname{Re}\delta>\delta(p)$, the Riesz means of order δ and type m are bounded on $H^p(R^N)$. In other words, Riesz means of different types are equivalent summability methods.

The approximation properties of the Riesz means of Fourier series and integrals have been the subject of several investigations (for example, see [1, 6]). In particular, in [5] Oswald proved that if f is in $H^p(R)$, $0<p\leqslant 1$, and if $\delta>1/p-1$, then

$$\|f-R_s^{\delta,2}*f\|_{H^p(R)}\leqslant c\omega_1(s,f,H^p(R))$$

(see also [12] for a related result on the approximation by Cesàro means of functions in $H^p(U)$). In this section we shall extend the above inequality to Hardy spaces of several variables. As we shall see this result is a quite easy consequence of the Jackson inequality in Theorem 4.1, and the Hörmander multiplier theorem for Hardy spaces.

Theorem 5.1. *If f is in $H^p(R^N)$, $0<p<+\infty$, m is a positive integer, and if* $\operatorname{Re}\delta$ *is greater than the critical index $\delta(p)$, then for every $s>0$ we have* $\|f-R_s^{\delta,m}*f\|_{H^p(R^N)}\leqslant c\omega_m(s,f,H^p(R)^N))$.

Proof. Since the decomposition of a (tempered distribution) f in $H^p(R^N)$ into a sum of (boundary values of) functions holomorphic in tube domains of C^N in Theorem 1.1 is "smoothness-preserving," we can suppose without loss of generality that the Fourier transform of f has support contained in $\overline{(R_+)^N}$ (i.e., f is in $H^p(D^N)$). Let $\Phi_s * f$ be defined as in Theorem 3.1, and let d be a (large) positive number. Then

$$\begin{aligned}\|f - R_s^{\delta,m} * f\|_{H^p(R^N)} &\\ \leqslant \operatorname{Max}\{1, 2^{1/p-1}\}(\|(f - \Phi_{ds} * f) - R_s^{\delta,m} * (f - \Phi_{ds} * f)\|_{H^p(R^N)} &\\ + \|\Phi_{ds} * f - R_s^{\delta,m} * \Phi_{ds} * f\|_{H^p(R^N)}).\end{aligned}$$

By Theorem 3.1 and the fact that the means $\{R_s^{\delta,m} * f\}$ are bounded on $H^p(R^N)$, we have

$$\begin{aligned}\|(f - \Phi_{ds} * f) - R_s^{\delta,m} * (f - \Phi_{ds} * f)\|_{H^p(R^N)} &\leqslant c\|f - \Phi_{ds} * f\|_{H^p(R^N)}\\ &\leqslant c\omega_m(s, f, H^p(R^N)).\end{aligned}$$

Denote by h the vector $(1, \ldots, 1) \in R^N$. Then

$$\begin{aligned}(\Phi_{ds} * f - R_s^{\delta,m} * \Phi_{ds} * f)\hat{}(\xi) &= \frac{(1-(1-|s\xi|^m)_+^\delta)}{(e^{2\pi i h\cdot s\xi}-1)^m}\,\phi(ds\xi)(e^{2\pi i s h\cdot\xi}-1)^m \hat{f}(\xi)\\ &= m(s\xi)(\Delta_{sh}^m f)\hat{}(\xi).\end{aligned}$$

A moment's reflection shows that in $(R_+)^N$ the function m is infinitely differentiable and satisfies the assumptions of the Hörmander multiplier theorem for Hardy spaces. Thus

$$\begin{aligned}\|\Phi_{ds} * f - R_s^{\delta,m} * \Phi_{ds} * f\|_{H^p(R^N)} \leqslant c\|\Delta_{sh}^m f\|_{H^p(R^N)} &\\ \leqslant c\omega_m(s, f, H^p(R^N)),\end{aligned}$$

and the theorem is proved. ∎

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